Robust synchronization

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The new open-plus-close-loop (OPCL) method of control complex dynamic systems developed by Jackson and Grosu [Physica D **85**, 1 (1995)] is used for synchronization of two identical oscillators. The synchronization is efficient for a very large level of noise. Numerical examples are given for two Lorenz systems and two logistic maps. The coupling is analytically justified. [S1063-651X(97)05809-1]

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There is a lot of work done on driving linear and nonlinear systems [1], but it was usually used just to obtain different types of dynamics [2,3] including chaotic behavior [4]. The idea of using a specific driving with the aim to obtain a desired behavior of a nonlinear system has been proposed by Hubler and Luscher [5] and then has been developed and optimized [6]. Control of complex dynamic systems has been the subject of a considerable interest during the past few years [7,8]. The synchronization is initiated by Pecora and Carroll [9] and was experimentally implemented by Cuomo and Oppenheim [10]. Now several different methods are known for coupling systems together in order to synchronize [11,12]. The present status, new improvements, and applications are presented in recent papers [13,14].

Recently, Jackson and Grosu [15] have developed a new powerful method of control: the open-plus-close-loop (OPCL) method. It can be applied to any model-based system

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^n \tag{1}$$

that can be drived. The driving term $\mathbf{D}(t)$ involves the sum of two actions;

$$\mathbf{D}(t)=\mathbf{H}(\mathbf{g},d\mathbf{g}/dt,t)+\mathbf{K}(g,\mathbf{x},t). \tag{2}$$

The open-loop action (Hubler action) [6] is

$$\mathbf{H}(\mathbf{g},d\mathbf{g}/dt,t)=d\mathbf{g}/dt-\mathbf{F}(\mathbf{g},t) \tag{3}$$

and the special linear feedback (closed-loop) [15] is

$$\mathbf{K}(\mathbf{g},\mathbf{x},t)=\left(\frac{d\mathbf{F}}{d\mathbf{g}}-\mathbf{A}\right)[\mathbf{g}(t)-\mathbf{x}(t)], \tag{4}$$

where $\mathbf{g}(t)\in\mathbb{R}^n$ is an arbitrary smooth function (the goal dynamics) and $\mathbf{A}$ is a constant matrix with negative real part eigenvalues. It was proved [15] that the driven system

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x},t)+S(t)\mathbf{D}(t) \tag{5}$$

[where $S(t)=0$ or 1 as a switch] assures $\mathbf{x}(t)\rightarrow\mathbf{g}(t)$ for any smooth $\mathbf{g}(t)$. So, in principle, any two oscillators can be synchronized. A Lorenz system can be drived to oscillate as a Rossler one or the inverse, but the driving term $\mathbf{D}(t)$ can be large. This paper presents applications of the OPCL method for synchronization of two (or several) identical oscillators. A comment was made [8] about Hubler action that ''the term is large and convergence is not assured.'' If the goal dynamics $\mathbf{g}(t)$ is different from the dynamics of the original system (1) (Lorenz vs Rossler, for example) then the term given by Eq. (3) can be large [16]. If the goal dynamics is the dynamics of a identical system or is the prerecorded dynamics [17] of the same system, then the term (3) is zero. The specific form of the driving $\mathbf{K}(\mathbf{g},\mathbf{x},t)$ (4) assures the convergence $\mathbf{x}(t)\rightarrow\mathbf{g}(t)$ [15] and in addition it can be arbitrarily small if $\mathbf{x}(0)-\mathbf{g}(0)$ is small enough. So, here are the answers to the both parts of the above-mentioned comment [8].

I have to emphasize that the freedom of choosing $\mathbf{A}$ (the only condition is to have eigenvalues with a negative real part) can be used in order to simplify the term $\mathbf{K}(\mathbf{g},\mathbf{x},t)$ (4). So, if dF_i/dg_k is constant, we can take $A_{ik}=dF_i/dg_k$ and $\mathbf{K}(\mathbf{g},\mathbf{x},t)$ is simpler. The driving term (2)–(4) assures the convergence for $\mathbf{x}(0)-\mathbf{g}(0)$ small enough. For particular systems it can be proved (if it can be found to be a Liapunov function) that the systems can be synchronized for any or resonable initial difference $\mathbf{x}(0)-\mathbf{g}(0)$ [15,18]. The OPCL method was previously used in migration control [18] for a Chua circuit that simulates a smart pacemaker. The above strategy is applied in the following for two identical Lorenz systems.

Let us consider the master system:

$$\dot{X}=s(Y-X), \quad \dot{Y}=rX-Y-XZ, \quad \dot{Z}=XY-bZ, \tag{6}$$

where $(s,r,b)=(16,45.6,4)$. The matrix (dF_i/dg_k) with $\mathbf{g}=(X,Y,Z)$ for Eq. (6) is

$$\begin{pmatrix} -s & s & 0 \\ r-Z & -1 & -X \\ Y & X & -b \end{pmatrix}.$$

This matrix has four variable terms. So K has to have at least four terms corresponding to these terms. The other constant terms of K can be zero by an appropriate choice of terms of A. This means that K will contain just one term (the simplest

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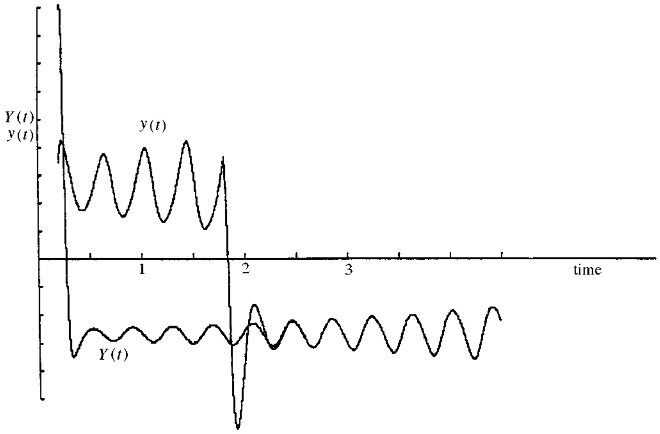

FIG. 1. $Y(t)$ and $y(t)$ described by Eqs. (6) and (8), respectively; initial conditions $X(0)=Y(0)=Z(0)=1$; $x(0)=y(0)=z(0)=-5$. $S=0$ for the first 1850 time steps, $S=1$ afterwards. Time step$=0.001$. $p=-60$. There is no noise.

form) if the nonlinearity in the system is represented just by one term of one variable (see the example for the Duffing oscillator below).

In addition A has to have a parameter, p, that can be adjusted in order that its eigenvalues have a negative real part. The parameter p is set in one position of variable terms in (dF_i/dg_k) in order to keep the number of terms in K to no more than four (for this case). Now, the matrix is

$$\mathbf{A}=\begin{pmatrix} -s & s & 0 \\ r+p & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \tag{7}$$

and the drived (slave) system is

$$\dot{x}=s(y-x),$$
$$\dot{y}=rx-y-xz+S(t)[X(z-Z)+(Z+p)(x-X)],$$
$$\dot{z}=xy-bz+S(t)[-X(y-Y)-Y(x-X)], \tag{8}$$

where p is a parameter that has to be determined in order to have $(x,y,z)\rightarrow(X,Y,Z)$. $S(t)=0$ or 1 as a switch.

The matrix (7) has eigenvalues with a negative real part if

$$p<1-r. \tag{9}$$

Let us note $\mathbf{u}=(u_1,u_2,u_3)=(x-X,y-Y,z-Z)$. For $\mathbf{u}$ the system can be obtained

$$\dot{u}_1=s(u_2-u_1), \quad \dot{u}_2=(r+p)u_1-u_2-u_1u_3,$$
$$\dot{u}_3=-bu_3+u_1u_2.$$

Choosing the Liapunov function $L=1/2(u_1^2+u_2^2+u_3^2)$, we have

$$\frac{dL}{dt}=-[su_1^2+u_2^2-(r+s+p)u_1u_2]-bu_3^2.$$

The condition $dL/dt<0$ gives us

$$-\sqrt{4s}-r-s<p<\sqrt{4s}-r-s. \tag{10}$$

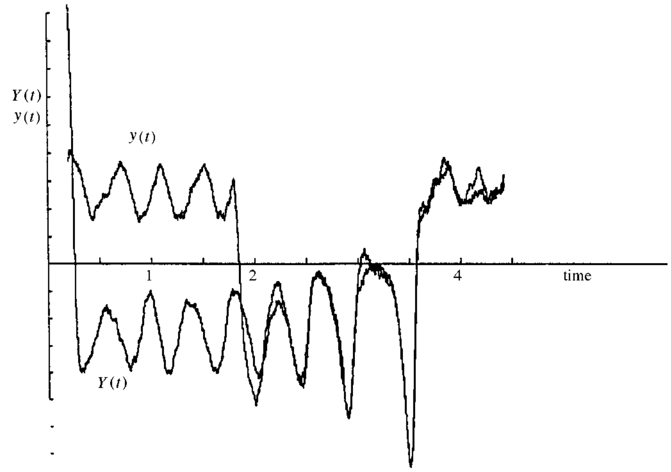

FIG. 2. The same as in Fig. 1, but each equation of Eqs. (6) and (8) is completed additively with a noise term (14) $C=500$.

So for any p given by Eq. (10) and any initial conditions of Eqs. (6) and (8), $u\rightarrow\mathbf{0}$, it means that $(x,y,z)\rightarrow(X,Y,Z)$. Both conditions (9) and ((10)) assure $(x,y,z)\rightarrow(X,Y,Z)$, but while condition (9) is for small enough $[x(0)-X(0),y(0)-Y(0),z(0)-Z(0)]$, condition (10) is for any initial conditions of the systems (6) and (8). This can be verified numerically (see Fig. 1). If the initial conditions are very different then the driving term can be large but once the coupling is set on the difference is smaller and smaller. If we can do just small drivings we have to wait until the difference between the two states is small and then to set $S=1$. For chaotic systems with one attractor and for any initial conditions of the true systems sooner or later will appear a situation when the two systems are close enough in order to put $S=1$. For chaotic systems with many attractors and if the dynamics are on different attractors, the method works too, but the driving term can be large for a short time [18].

The method can be applied to the synchronization of two drived Duffing oscillators. Kapitaniak [11] successfully synchronized two identical oscillators using a simple feedback:

$$\ddot{X}+a\dot{X}+X^3=B\cos t, \tag{11}$$

$$\ddot{x}+a\dot{x}+x^3=B\cos t+K(X-x), \tag{11$'$}$$

with $a=0.1$, $B=10$ and $0.01<K<0.1$ determined numerically. The present method can do the same synchronization:

$$\ddot{X}+a\dot{X}+X^3=B\cos t, \tag{12}$$

$$\ddot{x}+a\dot{x}+x^3=B\cos t+(3X^2-p)(x-X), \tag{13}$$

for any $a>0$ and $p>0$ (this result can be proved just by subtracting the above relations and keeping only linear terms in $u=x-X$). It can be observed that the above coupling is practical. A direct application of OPCL [15] gives a driving in velocity equation that is hard to realize. This is the main result obtained by a proper choice of the matrix A.

In addition, if we add a noise term

$$C\nearrow, \tag{14}$$

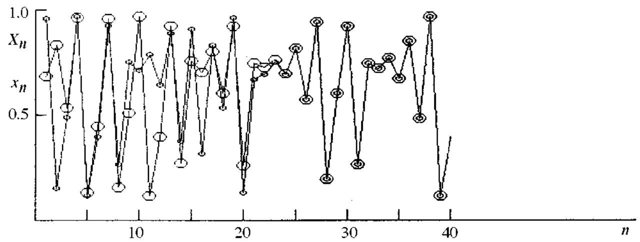

FIG. 3. $-X_n$ [described by Eq. (15)] and x_n [described by Eq. (16)] vs n, respectively. $S=0$ for the first 20 iterations. $X_1=0.77$, $x_1=0.55$, $c=3.88$, $p=0.1$. There is no noise. (X_n,n) is marked by a big circle and (x_n,n) by a small one.

where $\nearrow$ is a random number $\in(-1,1)$, in each equation of Eqs. (6) and (8), we still have synchronization for very large C (see Fig. 2). For $C=500$ the slave system follows the master system in a reasonable manner; so it looks robust.

It seems that is not so evident to write a control for discrete systems equivalent with Eqs. (2)–(5). Nevertheless we can synchronize two logistic dynamics.

The master system

$$X_{n+1}=cX_n(1-X_n) \tag{15}$$

and the slave (driven) system is

$$x_{n+1}=cx_n(1-x_n)+S[p-c(1-2X_n)](x_n-X_n), \tag{16}$$

with $c=3.88$ and $|p|<1$.

With $u_n=x_n-X_n$ from Eqs. (15) and (16) we have

$$u_{n+1}=pu_n-cu_n^2. \tag{17}$$

A Liapunov function $L(u)=u^2$ and the condition [19]

$$L(u_{n+1})-L(u_n)<0 \tag{18}$$

gives the result

$$\frac{p-1}{c}<u_n<\frac{p+1}{c} \tag{19}$$

in order to have $u_n\rightarrow 0$. Relation (19) gives the dimension of the basin of entrainment. So for any $u_n=x_n-X_n$, respecting Eq. (19), the sinchronization is obtained (Fig. 3). If we can manage just small drivings $D_n=S[p-c(1-2X_n)](x_n-X_n)$, we have to wait until x_n-X_n is small enough and then we can put $S=1$. Once the coupling is set on the driving decreases. In addition, the synchronization is obtained even if Eq. (16) is modified like (see Fig. 4)

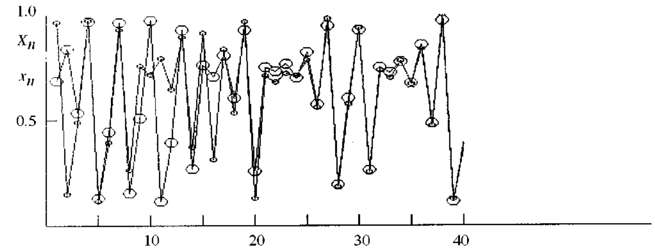

FIG. 4. $-X_n$ [described by Eq. (15)] and x_n [described by Eq. (20)] vs n with $a_1=0.04$, $a_2=0.04$; p, c, S are the same as in Fig. 3.

$$x_{n+1}=cx_n(1-x_n)+S\{[a_1\nearrow+p-c(1-2X_n)] \times(x_n-X_n)+a_2\nearrow\}, \tag{20}$$

where $\nearrow$ is a uniform random number between -1 and 1. This result led us to label this synchronization as robust.

It seems that the above results give a partial answer to the unpredictibility problem of chaotic systems: if we have a previous recording of a dynamics we can reach it by synchronization even in a noisy environment.

In conclusion, this paper presents a new method of synchronization based on the OPCL method of control [15]. The synchronization is realized with a precise coupling—so it does not need any trial and error, physical intuition, empirical determination of proportionality factor, calculation of conditional Liapunov exponents, etc. In addition, the synchronization is still obtained in a high level noise. There are no limitations on the dimensions of the systems.

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